Seat No.: _____ Enrolment No.______

GUJARAT TECHNOLOGICAL UNIVERSITY

B.E. Sem-I Examination January 2010

Subject code: 110008 **Subject Name: Mathematics – I**

Date: 11 / 01 /2010 **Time: 11.00 am – 02.00 pm**

Total Marks: 70

Instructions:

1. **Attempt all questions.**
2. **Make suitable assumptions wherever necessary.**
3. **Figures to the right indicate full marks.**

Q1. **(a)** **(i)** Find the value of *k* so that the function given below is continuous at a given point *x=2*. **02**

$$f(x)=\begin{cases}\dfrac{2^{x+2}-16}{4^{x}-16}, & x\neq 2\\ k, & x=2\end{cases}$$

(ii) State Sandwich theorem and using it find $\lim_{x\to 0} g(x)$ if $3-x^3\le g(x)\le 3\sec x$ for all x. **02**

(b) **(i)** If $f(x)$ and $g(x)$ are continuous functions for $0\le x\le 1$, could $f(x)/g(x)$ possibly be discontinuous at a point in the interval $[0,1]$? Give reasons for your answer. **02**

(ii) If $f:(a,b)\to\Re$ is differentiable at $c\in(a,b)$, then show that $\lim_{h\to 0^+}\dfrac{f(c+h)-f(c-h)}{2h}$ exists and equals $f'(c)$. Is the converse true? **02**

(c) **(i)** Using Mean Value Theorem, Prove $0<\dfrac{1}{x}\log\left(\dfrac{e^x-1}{x}\right)<1,\ for\ x>0.$ **03**

(ii) For what values of a,m and b does the function **03**

$$f(x)=\begin{cases}3, & x=0\\ -x^2+3x+a, & 0<x<1\\ mx+b, & 1\le x\le 2\end{cases}$$

satisfy the hypothesis of the Mean Value Theorem on the interval $[0,2]$.

Q2. **(a)** **(i)** Find the area of the region between the *x-axis* and the graph of $f(x)=x^3-x^2-2x,\ -1\le x\le 2.$ **02**

(ii) Using Fundamental Theorem of Calculus find dy/dx if $y=\int_1^{x^2}\cos t\,dt$. **02**

(iii) Evaluate the integral $\int_0^{\infty}\dfrac{dx}{x^2+1}$. **02**

(b) **(i)** Find the absolute maximum and minimum values of the function on the given interval $f(t)=|t-5|,\ 4\le t\le 7$. **03**

(ii) Find the Taylor's series expansion of $f(x)=x^3-2x+4,\quad a=2.$ **02**

(iii) The geometric mean of two positive numbers a and b is the number $\sqrt{ab}$. Show that the value of c in the conclusion of the Mean Value Theorem for $f(x) = 1/x$ on an interval of positive numbers $[a,b]$ is $c = \sqrt{ab}$. **02**

OR

(b) (i) Test the convergence or divergence of the following series **(ANY TWO)** **04**

a. $\sum_{n=0}^{\infty} \frac{2^n - 1}{3^n}$ b. $\sum_{n=1}^{\infty} \frac{n}{e^{-n}}$ c. $\sum_{n=0}^{\infty} n!(x-4)^n$

(ii) Using Riemann Sum show that $\int_a^b x \, dx = \frac{1}{2}\left(b^2 - a^2\right)$ **03**

Q3. (a) Suppose that $w = f(x,y)$, $x = g(r,s)$ *and* $y = h(r,s)$ then write the chain rule for $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$. Also evaluate $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ in terms of r and s if $w = x + 2y + z^2$, $x = r/s$, $y = r^2 + \ln s$, $z = 2r$. **05**

(b) (i) Show that $\int_{-\infty}^{\infty} f(x)\, dx$ may not equal to $\lim_{b\to\infty} \int_{-b}^{b} f(x)\, dx$. **02**

(ii) If $u = \tan^{-1}\left(\frac{x^2 + y^2}{x - y}\right)$ show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\sin 2u$ **03**

(c) Find the length of the curve $y = \int_0^x \sqrt{\cos 2t}\, dt$ from $x = 0$ to $x = \pi/4$. **04**

OR

Q3. (a) Let $w = f(x,y,z)$ be a function of three independent variables, write the formal definition of the partial derivative for $\partial f/\partial z$ at (x_0, y_0, z_0). Using this definition find $\partial f/\partial z$ at $(1,2,3)$ for $f(x,y,z) = x^2yz^2$. **05**

(b) (i) Show that **03**

$$f(x,y) = \begin{cases} \frac{2xy}{x^2 + y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

is continuous at every point except at the origin.

(ii) Find dw/dt if $w = xy + z$, $x = \cos t$, $y = \sin t$, $z = t$. **02**

(c) Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 2$ and $x = 0$ about the line $y = 2$. **04**

Q4. (a) (i) Evaluate the integral $\int_0^1 \int_0^{\sqrt{1-y^2}} (x^2 + y^2)\, dx\, dy$ **03**

(ii) Find the volume of the region that lies under the paraboloid $z = x^2 + y^2$ and above the triangle enclosed by the lines $y = x, x = 0$ *and* $x + y = 2$ in the $xy-$plane. **04**

(b) Find the equations for tangent plane and normal line at the point $(1,1,1)$ on the surface $x^2 + y^2 + z^2 = 3$. **03**

(c) Find the area of the region that lies inside the cardioid $r = 1 + \cos\vartheta$ and outside the circle $r = 1$. **04**

OR

Q4. (a) Evaluate $\int_0^1 \int_0^{\sqrt{z}} \int_0^{2\pi} \left(r^2 \cos^2\vartheta + z^2\right) r\, d\vartheta\, dr\, dz$. **04**

(b) (i) Integrate $f(x,y) = \dfrac{\ln(x^2 + y^2)}{\sqrt{x^2 + y^2}}$ over the region $1 \le x^2 + y^2 \le e$ by changing to polar coordinates **04**

(ii) Find the derivative of $f(x,y,z) = x^3 - xy^2 - z$ at $P_0(1,1,0)$ in the direction of $\vec{v} = 2\vec{i} - 3\vec{j} + 6\vec{k}$. **02**

(c) Find the volume of the prism whose base is the triangle in xy - plane bounded by the x – axis and the line $y = x$ and $x = 1$ and whose top lies in the plane $z = f(x,y) = 3 - x - y$.. **04**

Q5. (a) Integrate $f(x,y,z) = x + \sqrt{y} - z^2$ over the path $C = C_1 \cup C_2$ from $(0,0,0)$ *to* $(1,1,1)$ with **05**

$C_1 : r(t) = t\,\vec{i} + t^2\,\vec{j},\ 0 \le t \le 1$

$C_2 : r(t) = \vec{i} + \vec{j} + t\,\vec{k},\ 0 \le t \le 1$

(b) State Green's theorem and also evaluate the integral $\oint_C (6y + x)dx + (y + 2x)dy$ where C : The circle $(x-2)^2 + (y-3)^2 = 4$. **05**

(c) Trace the curve $r^2 = a^2 \cos 2\vartheta$. **04**

OR

Q5. (a) Use Green's theorem to evaluate the integral $\oint_C \left(y^2 dx + x^2 dy\right)$ where C: The triangle bounded by $x = 0$, $x + y = 1$, $y = 0$. **05**

(b) Find the flux of $F = yz\,\vec{j} + z^2\,\vec{k}$ outward through the surface S cut from the cylinder $y^2 + z^2 = 1,\ z \ge 0,$ by the planes $x = 0$ and $x = 1$. **05**

(c) Use Stoke's theorem to evaluate $\int_C F.dr$ if **04**

$F = (x+y)\,\vec{i} + (2x - z)\,\vec{j} + (y + z)\,\vec{k}$ and C is the boundary of the triangle $(2,0,0)$, $(0,3,0)$ and $(0,0,6)$.

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