Seat No.: ________ Enrolment No.___________

GUJARAT TECHNOLOGICAL UNIVERSITY

BE - SEMESTER– 1st / 2nd • EXAMINATION – WINTER 2013

Subject Code: 110015 **Date: 17-12-2013**

Subject Name: VECTOR CALCULUS AND LINEAR ALGEBRA

Time: 10:30 am – 01:30 pm **Total Marks: 70**

Instructions:

1. **Attempt any five questions.**
2. **Make suitable assumptions wherever necessary.**
3. **Figures to the right indicate full marks.**

Q.1 (a) (i) Show that the differential form under the integral of $I = \int_{(0,-1,1)}^{(2,4,0)} e^{x-y+z^2}(dx - dy + 2z dz)$ is exact in space and evaluate the integral. **5**

(ii) A parametric representation of the surface is given. Identify and sketch the surface: **2**
$\bar{r}(u,v) = a\cos u\,\hat{i} + a\sin u\,\hat{j} + v\,\hat{k}$, where u, v vary in the rectangle $R: 0 \le u \le 2\pi,\ -1 \le v \le 1$.

(b) For which values of 'a' will the following system have no solution? Exactly one solution? Infinitely many solutions? **4**

$x + 2y - 3z = 4$
$3x - y + 5z = 2$
$4x + y + (a^2 - 14)z = a + 2$.

(c) Let A be the matrix **3**

$\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$, $p_1(x) = x^2 - 9$, $p_2(x) = x + 3$, and $p_3(x) = x - 3$,

show that $p_1(A) = p_2(A)p_3(A)$.

Q.2 (a) (i) Verify Gauss divergence theorem for $\bar{F} = 7x\hat{i} - z\hat{k}$ over the sphere $x^2 + y^2 + z^2 = 4$. **5**

(ii) Find the directional derivative of $f(x, y, z) = xyz$ at the point $p:(-1,1,3)$ in the direction of the vector $\bar{a} = \hat{i} - 2\hat{j} + 2\hat{k}$. **2**

(b) **4**

Find the inverse of a matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$ using Row operations.

(c) Determine whether the set of all polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ for which a_0, a_1, a_2 and a_3 are integers, is a subspace of P_3. **3**

Q.3 (a) (i) Show that the set of all 2×2 matrices of the form $\begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix}$ with addition defined by $\begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix} + \begin{bmatrix} c & 1 \\ 1 & d \end{bmatrix} = \begin{bmatrix} a+c & 1 \\ 1 & b+d \end{bmatrix}$ and scalar multiplication defined by $k\begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix} = \begin{bmatrix} ka & 1 \\ 1 & kb \end{bmatrix}$ is a vector space. **5**

(ii) Find the area of the papallelogram determined by the vectors $\overline{u} = (2,3,0)$, $\overline{v} = (-1,2,-2)$. **2**

(b) Using Green's theorem, evaluate the line integral $\oint_C (\sin y\,dx + \cos x\,dy)$ counter clockwise, where C is the boundary of the triangle with vertices $(0,0)$, $(\pi,0)$, $(\pi,1)$. **4**

(c) The velocity vector $\overline{v} = \overline{r}'(t) = x^3\hat{k}$ of a fluid motion is given. Is the flow irrotational? Incompressible? Find the path of the particle. **3**

Q.4 (a) (i) Let $p_1 = 1+x$, $p_2 = 1+x^2$ and $p_3 = x+x^2$. Show that the set $S = \{p_1, p_2, p_3\}$ is a basis for $\boldsymbol{P}_2$. Find the coordinate vector of $p = 2-x+x^2$ with respect to S. **5**

(ii) Use appropriate identities, where required, to determine which of the following sets of vectors in $F(-\infty,\infty)$ are linearly dependent: **2**

(1) x, $\cos x$ (2) $\cos 2x$, $\sin^2 x$, $\cos^2 x$.

(b) Find the rank and nullity of the matrix $A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$. **4**

(c) Use Cramer's rule to solve the system **3**

$x_1 - 3x_2 + x_3 = 4$, $2x_1 - x_2 = -2$, $4x_1 - 3x_3 = 0$.

Q.5 (a) (i) Let W be the space of P^5 spanned by the vectors $\overline{v}_1 = (1,4,5,6,9)$, $\overline{v}_2 = (3,-2,1,4,-1)$, $\overline{v}_3 = (-1,0,-1,-2,-1)$, $\overline{v}_4 = (2,3,5,7,8)$. Find a basis for the orthogonal complement of W. **5**

σ

(ii) Sketch the unit circle in an xy-coordinate system in R^2 using the Euclidean inner product $\langle \overline{u}, \overline{v} \rangle = \frac{1}{4}u_1v_1 + \frac{1}{16}u_2v_2.$ **2**

(b) Find the least squares solution of the linear system A**X**=**b** given by $2x-2y=2$, $x+y=-1$,$3x+y=1$. Also find the orthogonal projection of **b** on the column space of A. **4**

(c) Let R^2 have the Euclidean inner product. Use Gram-Schmidt process to transform the basis vectors $\overline{u}_1 = (1,-3)$, $\overline{u}_2 = (2,2)$ into an orthonormal basis. **3**

Q. 6 (a) Find a matrix P that diagonalizes $A=\begin{bmatrix}-1 & 4 & -2\\ -3 & 4 & 0\\ -3 & 1 & 3\end{bmatrix}$, and determine $P^{-1}AP$. **7**

(b) (i) Find the geometric and algebraic multiplicity of each eigen value of $A=\begin{bmatrix}2 & 0\\ 1 & 2\end{bmatrix}$. **2**

(ii) Let $\overline{u}=(u_1,u_2)$, $\overline{v}=(v_1,v_2)$ be vectors in R^2. Verify that the weighted Euclidean inner product $\langle\overline{u},\overline{v}\rangle=3u_1v_1+5u_2v_2$ satisfies the four inner product axioms. **2**

(c) Given the quadratic equation $x^2-16y^2+8x+128y=256$. A translation will put the conic in standard position. Name the conic and give its equation in the translated coordinate system. **3**

Q.7 (a) (i) Find the standard matrix for the stated composition of linear operators on R^2: **5**

(a) A rotation of 60°, followed by an orthogonal projection on the x-axis, followed by a reflection about the line $y=x$.

(b) A dilation with factor k=2, followed by a rotation of 45°, followed by a reflection about the y-axis.

(ii) Determine whether the function $T:V\to R$, where V is an inner product space, and $T(\boldsymbol{u})=\|\boldsymbol{u}\|$, is a linear transformation. Justify your answer. **2**

(b) Let $T:R^2\to R^3$ be the linear transformation defined by $T\left(\begin{bmatrix}x_1\\ x_2\end{bmatrix}\right)=\begin{bmatrix}x_2\\ -5x_1+13x_2\\ -7x_1+16x_2\end{bmatrix}$. Find the matrix for the transformation T with respect to the bases $B=\{\overline{u}_1,\overline{u}_2\}$ for R^2 and $B'=\{\overline{v}_1,\overline{v}_2,\overline{v}_3\}$ for R^3, where **4**

$$\overline{u}_1=\begin{bmatrix}3\\ 1\end{bmatrix},\ \overline{u}_2=\begin{bmatrix}5\\ 2\end{bmatrix},\ \overline{v}_1=\begin{bmatrix}1\\ 0\\ -1\end{bmatrix},\ \overline{v}_2=\begin{bmatrix}-1\\ 2\\ 2\end{bmatrix},\ \overline{v}_3=\begin{bmatrix}0\\ 1\\ 2\end{bmatrix}.$$

(c) Show that the linear operator $T:R^2\to R^2$ defined by the equations **3**

$$\begin{aligned}w_1&=x_1+2x_2\\ w_2&=-x_1+x_2\end{aligned}$$

is one-to-one, and find $T^{-1}(w_1,w_2)$.

****