Seat No.: ________  Enrolment No.___________

GUJARAT TECHNOLOGICAL UNIVERSITY
BE SEMESTER 1st / 2nd (NEW) EXAMINATION WINTER 2016

Subject Code: 2110015  Date:20/01/2017
Subject Name: Vector Calculus and Linear Algebra
Time:10:30 AM TO 1:30 PM  Total Marks: 70

Instructions:
1. Question No. 1 is compulsory. Attempt any four out of remaining Six questions.
2. Make suitable assumptions wherever necessary.
3. Figures to the right indicate full marks.

Q.1  Objective Question (MCQ)  Mark
(a) Choose the appropriate answer for the following MCQs.  07

1. Rank of $3 \times 3$ invertible matrix is
   a) 1  b) 2  c) 3  d) 4

2. Let $A$ be a Skew- Hermitian matrix then $A =$ __________
   a) $A^T$  b) $(\bar{A})^T$  c) $-A^T$  d) $-(\bar{A})^T$

3. The set $S = \{1, x, x^2, x^3\}$ span which of the following?
   a) $P_1$  b) $R$  c) $R^3$  d) $M_{33}$

4. Let $u = (1, -2)$ be a vector in $R^2$ with the Euclidean inner product, then $\|u\|$ is
   a) 1  b) 5  c) $\sqrt{5}$  d) $\sqrt{3}$

5. If $A$ is a matrix with 6 columns and $\text{rank}(A) = 2$, then $\text{Nullity}(A)$ is
   a) 2  b) 4  c) 0  d) None of these

6. If $\lambda_1 = 2, \lambda_2 = 6$ are the eigen values of the matrix $A$, then the eigen values of $A^T$ are
   a) 2 & 6  b) $\frac{1}{2}$ & $\frac{1}{6}$  c) 4 & 36  e) None of these

7. The product of the eigen values of matrix $A = \begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$ is,
Choose the appropriate answer for the following MCQs.

1. Let $I : V \rightarrow V$ be an identity operator, then $\ker (I)$ is,

   a) $V$  
   b) $\{0\}$  
   c) 0  
   d) None of these

2. The mapping $T : R^3 \rightarrow R^3$ defined by $T(x, y, z) = (x, y, 0)$ is called as

   a) Projection  
   b) Reflection  
   c) Rotation  
   d) Magnification

3. If $f_1 = x$ and $f_2 = \sin x$, then Wronskian $\begin{pmatrix} \pi \\ 2 \end{pmatrix} = \ldots$

   a) $\frac{\pi}{2}$  
   b) 1  
   c) 0  
   d) -1

4. If $\vec{r} = xi + yj + zk$, then divergence of $\vec{r}$ is

   a) 2  
   b) -2  
   c) 3  
   d) -3

5. The value of $\text{curl} (\text{grad} \phi)$, where $\phi = 2x^2 - 3y^2 + 4z^2$ is

   a) $4xi - 6yj + 8zk$  
   b) $4x - 6y + 8z$  
   c) 6  
   d) $\frac{\vec{r}}{0}$

6. The weighted Euclidean inner product $\langle u, v \rangle = 3u_1v_1 + 2u_2v_2$ is the inner product on $R^2$ generated by

   a) $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$  
   b) $\begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$  
   c) $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$  
   d) $\begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix}$

7. If $V$ is a finite-dimensional vector space, and $T : V \rightarrow V$ is a linear operator and $\ker (T) = \{0\}$, then

   a) $R(T) \neq V$  
   b) $T$ is one-to-one  
   c) $\text{Nullity} (T) \neq 0$  
   d) None of these

Q.2 (a) Convert the following matrix in to reduced row echelon form and hence find the rank of a matrix.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$$

(b) Solve the following system of equations by Gauss Elimination method

$$x_1 - 2x_2 + 3x_3 = -2$$
$$-x_1 + x_2 - 2x_3 = 3$$
$$2x_1 - x_2 + 3x_3 = -7$$
(c) Find the inverse of the matrix \( A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \) using Gauss Jordan Method.

(ii) For what choices of parameter the following system is consistent.
\[
\begin{align*}
&x_1 + x_2 + 2x_3 + x_4 = 1 \\
&x_1 + 2x_3 = 0 \\
&2x_1 + 2x_2 + 3x_3 = \lambda \\
&x_2 + x_3 + 3x_4 = 2\lambda
\end{align*}
\]

Q.3 (a) Consider the vectors \( u = (1,2,-1) \) and \( v = (6,4,2) \) in \( \mathbb{R}^3 \). Show that \( w = (9,2,7) \) is a linear combination of \( u \) and \( v \).

(b) Determine a basis for and the dimension of the solution space of the homogeneous system:
\[
\begin{align*}
&2x_1 + 2x_2 - x_3 + x_4 = 0 \\
&-x_1 - x_2 + 2x_3 - 3x_4 + 3x_5 = 0 \\
&x_1 + x_2 - 2x_3 - x_4 = 0 \\
&x_3 + x_4 + x_5 = 0
\end{align*}
\]

(c) Show that the set of all pairs of real numbers of the form \( (1,x) \) with the operations \( (1,x) + (1,y) = (1, x+y) \) and \( k(1,x) = (1,ky) \) is a vector space.

Q.4 (a) Sketch the unit circle in an \( xy \)-coordinate system in \( \mathbb{R}^2 \) using

1. The Euclidean inner product \( \langle u,v \rangle = u_1v_1 + u_2v_2 \).

2. The weighted Euclidean inner product \( \langle u,v \rangle = \frac{1}{9}u_1v_1 + \frac{1}{4}u_2v_2 \).

(b) Attempt the following.

1. Let \( u = (u_1,u_2) \) and \( v = (v_1,v_2) \) be vectors in \( \mathbb{R}^2 \). Verify that the weighted Euclidean inner product \( \langle u,v \rangle = 3u_1v_1 + 2u_2v_2 \) satisfies the four inner product axioms.

2. Let \( \mathbb{R}^4 \) have the Euclidean inner product. Find the cosine of the angle \( \theta \) between the vectors \( u = (4,3,1,-2) \) and \( v = (-2,1,2,3) \).

(c) Consider the vector space \( \mathbb{R}^3 \) with the Euclidean inner product. Apply the Gram-Schmidt process to transform the basis vectors \( u_1 = (1,1,1), u_2 = (0,1,1) \) \& \( u_3 = (0,0,1) \) into an orthogonal basis \( \{v_1,v_2,v_3\} \); then normalize the orthogonal basis vectors to obtain an orthonormal basis \( \{q_1,q_2,q_3\} \).

Q.5 (a) Let \( T : P_1 \rightarrow P_2 \) be the linear transformation defined by \( T(p(x)) = x(p(x)) \).

Find the matrix for \( T \) with respect to the standard bases \( B = \{u_1,u_2\} \) and \( B' = \{v_1,v_2,v_3\} \), where \( u_1 = 1, u_2 = x; v_1 = 1, v_2 = x, v_3 = x^2 \).

(b) Express \( \begin{bmatrix} 4+2i & 7 & 3-i \\ 0 & 3i & -2 \\ 5+3i & -7+i & 9+6i \end{bmatrix} \) as the sum of a Hermitian and a skew-Hermitian matrix.

(c) State the Dimension theorem for Linear Transformation and find the rank and nullity.
of $T_A$, where $T_A : \mathbb{R}^6 \to \mathbb{R}^4$ be multiplication by
\[
\begin{bmatrix}
-1 & 2 & 0 & 4 & 5 & -3 \\
3 & -7 & 2 & 0 & 1 & 4 \\
2 & -5 & 2 & 4 & 6 & 1 \\
4 & -9 & 2 & -4 & -4 & 7
\end{bmatrix}
\]

\[A = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix}\]

\[A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & -1 & 4 \\ -2 & 1 & 5 \end{bmatrix}\]

\[\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\]

\[\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\]

Q.6 (a) Determine the algebraic and geometric multiplicity of

(b) For the matrix $A = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix}$, find the nonsingular matrix $P$ and the diagonal matrix $D$ such that $D = P^{-1}AP$.

(c) Determine $A^{-1}$ by using Cayley-Hamilton theorem for the matrix

\[A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & -1 & 4 \\ -2 & 1 & 5 \end{bmatrix}\]

Hence find the matrix represented by

\[A^k - 5A^7 - A^6 + 37A^5 + A^4 - 5A^3 - 3A^2 + 41A + 3I\]

Q.7 (a) Show that $F = (y^2 - z^2 + 3yz - 2x)i + (3xz + 2xy) j + (3xy - 2xz + 2z) k$ is both solenoidal and irrotational.

(b) Find the work done when a force $F = (x^2 - y^2 + 2x)i - (2xy + y) j$ moves a particle in the $xy$-plane from $(0, 0)$ to $(1, 1)$ along the parabola $y^2 = x$. Is the work done different when the path is the straight line $y = x$?

(c) State Green’s theorem and use it to evaluate the integral $\int_C y^2 dx + x^2 dy$, where $C$ is the triangle bounded by $x = 0, x + y = 1, y = 0$. *************