

GUJARAT TECHNOLOGICAL UNIVERSITY

B.E. Sem-II Examination June 2010

Subject code: 110009

Subject Name: Mathematics-II

Date: 23 /06 /2010

Time: 02.30 pm – 05.30 pm

Total Marks: 70

Instructions:

1. Attempt all questions.
2. Make suitable assumptions wherever necessary.
3. Figures to the right indicate full marks.

- Q.1** (a) Attempt any two: **06**
- i. Solve the following system for x, y and z :
- $$-\frac{1}{x} + \frac{3}{y} + \frac{4}{z} = 30, \quad \frac{3}{x} + \frac{2}{y} - \frac{1}{z} = 9, \quad \frac{2}{x} - \frac{1}{y} + \frac{2}{z} = 10.$$
- ii. Find A^{-1} using row operations if $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.
- iii. Find the standard matrices for the reflection operator about the line $y = x$ on R^2 and the reflection operator about the yz -plane on R^3 .
- (b) Show that there is no line containing the points $(1,1)$, $(3,5)$, $(-1,6)$ and $(7,2)$. **03**
- (c) i. Find all vectors in R^3 of Euclidean norm 1 that are orthogonal to the vectors $u_1 = (1,1,1)$ and $u_2 = (1,1,0)$. **02**
- ii. Find the rank of the matrix $A = \begin{bmatrix} 2 & -1 & 3 \\ 4 & -2 & 6 \\ -6 & 3 & -8 \end{bmatrix}$ in terms of determinants. **02**
- iii. Is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ in row-echelon or reduced row-echelon form? **01**
- Q.2**
- (a) i. What conditions must b_1, b_2 and b_3 satisfy in order for $x_1 + 2x_2 + 3x_3 = b_1$, $2x_1 + 5x_2 + 3x_3 = b_2$, $x_1 + 8x_3 = b_3$ to be consistent? **04**
- ii. Is $T : R^3 \rightarrow R^3$ defined by $T(x, y, z) = (x + 3y, y, z + 2x)$ linear? Is it one-to-one, onto or both? Justify. **03**
- (b) i. Show that the set $S = \{e^x, xe^x, x^2e^x\}$ in $C^2(-\infty, \infty)$ is linearly independent. **02**
- ii. Check whether $V = R^2$ is a vector space with respect to the operations $(u_1, u_2) + (v_1, v_2) = (u_1 + v_1 - 2, u_2 + v_2 - 3)$ and $\alpha(u_1, u_2) = (\alpha u_1 + 2\alpha - 2, \alpha u_2 - 3\alpha + 3)$, $\alpha \in R$. **05**

OR

- (b) i. State only one axiom that fails to hold for each of the following sets W to be subspaces of the respective real vector space V with the standard operations: 05
- [A] $W = \{(x, y) \mid x^2 = y^2\}$, $V = \mathbb{R}^2$
- [B] $W = \{(x, y) \mid xy \geq 0\}$, $V = \mathbb{R}^2$
- [C] $W = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$, $V = \mathbb{R}^3$
- [D] $W = \{A_{n \times n} \mid Ax = 0 \Rightarrow x = 0\}$, $V = M_{n \times n}$
- [E] $W = \{f \mid f(x) \leq 0, \forall x\}$, $V = F(-\infty, \infty)$
- ii. Check whether $S = \{\sin(x+1), \sin x, \cos x\}$ in $C(0, \infty)$ is linearly independent. 02

Q.3

- (a) i. Determine whether the following polynomials span P_2 : 03
 $p_1 = 1 - x + 2x^2$, $p_2 = 5 - x + 4x^2$, $p_3 = -2 - 2x + 2x^2$.
- ii. Show that $S = \{1 - t - t^3, -2 + 3t + t^2 + 2t^3, 1 + t^2 + 5t^3\}$ is linearly independent in P_3 . 03
- (b) i. Find a standard basis vector that can be added to the set $S = \{(-1, 2, 3), (1, -2, -2)\}$ to produce a basis of \mathbb{R}^3 . 03
- ii. Determine whether b is in the column space of A , and if so, express b as a linear combination of the column vectors of A if 03
- $$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \text{ and } b = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}.$$
- (c) i. If A is an $m \times n$ matrix, what is the largest possible value for its rank. 01
- ii. Find the number of parameters in the general solution of $Ax = 0$ if A is a 5×7 matrix of rank 3. 01

OR

Q.3

- (a) i. Find basis and dimension of 03
 $W = \{(a_1, a_2, a_3, a_4) \in \mathbb{R}^4 \mid a_1 + a_2 = 0, a_2 + a_3 = 0, a_3 + a_4 = 0\}$.
- ii. Find a basis for the subspace of P_2 spanned by the vectors 03
 $1 + x, x^2, -2 + 2x^2, -3x$.
- (b) i. Reduce $S = \{(1, 0, 0), (0, 1, -1), (0, 4, -3), (0, 2, 0)\}$ to obtain a basis of \mathbb{R}^3 . 03
- ii. Find a basis for the row space of A and column space of A if 03
- $$A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 0 \end{bmatrix}.$$
- Also verify the dimension theorem for matrices.
- (c) Show that $S = \left\{ \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -1 & 2 \end{bmatrix} \right\}$ is a basis 02
for M_{22} .

Q.4

- (a) i. Compute $d(f, g)$ for $f = \cos 2\pi x$ and $g = \sin 2\pi x$ in $V = C[0,1]$ with inner product $\langle f, g \rangle = \int_0^1 f(x)g(x)dx$. **02**
- ii. Find a basis for the orthogonal complement of the subspace of R^3 spanned by the vectors $v_1 = (1, -1, 3)$, $v_2 = (5, -4, -4)$ and $v_3 = (7, -6, 2)$. **03**
- (b) i. Let $W = \text{span}\left\{\left(\frac{4}{5}, 0, \frac{-3}{5}\right), (0, 1, 0)\right\}$. Express $w = (1, 2, 3)$ in the form of $w = w_1 + w_2$, where $w_1 \in W$ and $w_2 \in W^\perp$. **03**
- ii. Define algebraic and geometric multiplicity. Show that $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$ is not diagonalizable. **03**
- (c) Show that P_3 and M_{22} are isomorphic. **03**

OR**Q. 4**

- (a) i. Let R^3 have the Euclidean inner product. Transform the basis $S = \{(1, 0, 0), (3, 7, -2), (0, 4, 1)\}$ into an orthonormal basis using the Gram-Schmidt process. **03**
- ii. For $U = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix}$ and $V = \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix}$ in M_{22} , define $\langle U, V \rangle = u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4$. For the matrices A and B , verify Cauchy-Schwarz inequality and find the cosine of the angle between them, if $A = \begin{bmatrix} 2 & 6 \\ 1 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$. **02**
- (b) i. Find the least squares solution of the linear system $AX = b$ and find the orthogonal projection of b onto the column space of A where $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -1 & 2 \end{bmatrix}$ and $b = \begin{bmatrix} 7 \\ 0 \\ -7 \end{bmatrix}$. **03**
- ii. Find the transition matrix from basis $B = \{(1, 0), (0, 1)\}$ of R^2 to basis $B' = \{(1, 1), (2, 1)\}$ of R^2 . **03**
- (c) For the matrix $A = \begin{bmatrix} \frac{1+i}{2} & \frac{1+i}{2} \\ \frac{1-i}{2} & \frac{-1+i}{2} \end{bmatrix}$, show that the row vectors form an orthonormal set in C^2 . Also, find A^{-1} . **03**

Q.5

- (a) i. For the basis $S = \{v_1, v_2, v_3\}$ of R^3 , where $v_1 = (1, 1, 1)$, $v_2 = (1, 1, 0)$ and $v_3 = (1, 0, 0)$, let $T : R^3 \rightarrow R^3$ be a linear transformation such that $T(v_1) = (2, -1, 4)$, $T(v_2) = (3, 0, 1)$, $T(v_3) = (-1, 5, 1)$. Find a formula for $T(x_1, x_2, x_3)$ and use it to find $T(2, 4, -1)$. **04**
- ii. Let $T_1 : M_{22} \rightarrow R$ and $T_2 : M_{22} \rightarrow M_{22}$ be the linear transformations given by $T_1(A) = \text{tr}(A)$ and $T_2(A) = A^T$. **02**
 Find $(T_1 \circ T_2)(A)$ where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.
- (b) Find a matrix P that diagonalizes $A = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$, and hence find A^{10} . Also, find the eigenvalues of A^2 . **04**
- (c) Let $T : R^4 \rightarrow R^3$ be the linear transformation given by $T(x_1, x_2, x_3, x_4) = (w_1, w_2, w_3)$ where $w_1 = 4x_1 + x_2 - 2x_3 - 3x_4$, $w_2 = 2x_1 + x_2 + x_3 - 4x_4$, $w_3 = 6x_1 - 9x_3 + 9x_4$. Find bases for the range and kernel of T . **04**

OR**Q.5**

- (a) Let $T : R^2 \rightarrow R^3$ be the linear transformation defined by $T(x_1, x_2) = (x_2, -5x_1 + 13x_2, -7x_1 + 16x_2)$. **04**
 Find the matrix for the transformation T with respect to the bases $B = \{(3, 1)^T, (5, 2)^T\}$ for R^2 and $B' = \{(1, 0, -1)^T, (-1, 2, 2)^T, (0, 1, 2)^T\}$ for R^3 .
- (b) i. Let $T : R^2 \rightarrow R^2$ be defined by $T(x, y) = (x + y, x - y)$. Is T one-one? If so, find formula for $T^{-1}(x, y)$. **04**
- ii. Find eigenvalues of $A = \begin{bmatrix} -420 & 1/2 & 576 \\ 0 & 0 & 0.6 \\ 0 & 0 & \sqrt{3} \end{bmatrix}$. Is A invertible? **01**
- (c) Translate and rotate the coordinate axes, if necessary, to put the conic $9x^2 - 4xy + 6y^2 - 10x - 20y = 5$ in standard position. Find the equation of the conic in the final coordinate system. **05**
